ON THE BUCKLING OF FLEXIBLE PLATES

PMM Vol.32, №4, 1968, pp.721-727

L.S.SRUBSHCHIK and V.A.TRENOGIN
(Rostov-on-Don and Moscow)
(Received October 25, 1967)

The problem considered is that of the behavior of a thin flexible plate of arbitrary shape under the influence of small transverse loading εQ and external boundary forces, the components of which are p and q along the x and y axes, respectively. In the case when the transverse loading is absent, $p = p_0$ and $q = q_0$ this problem has, as is well known, the trivial solution

$$w = 0, \quad F = {}^1/_2\,(p_0 x^2 + q_0 y^2)$$

where w is the deflection of the plate and F is a stress function. There exist one or several equilibrium forms of the plate near this solution for small deviations of p and q from p_0 and q_0 and for small transverse load. If p_0 and q_0 are not critical, then there exists a single equilibrium form, which depends analytically on the parameters mentioned above. A method of determining the solution for this case in the form of a power series in a small parameter was proposed by Polubarinova-Kochina [1] and was later substantiated by Vorovich [2].

The case of critical values of p_0 and q_0 is more complicated (the phenomenon of buckling of a plate is well known). The first investigation for a circular plate under the condition of radial symmetry ($p = q$) and without lateral load was carried out by Friedrichs and Stoker [3] who used a variational method to demonstrate the creation of a pair of new solutions as the critical value is exceeded. Quite recently, Berger and Fife [4] extended these results to the case of a plate of arbitrary shape without transverse load, under the assumption that the boundary forces depend on a single parameter. The solution was obtained by making use of a topological theorem on bifurcations due to Krasnosel'skii [5]. Much earlier, in 1955-1958, Vorovich [6] gave a qualitative analysis of the postbuckling behavior of plates and shells in an investigation of general problems of the nonlinear theory of shallow shells. In this study, he applied the theory of eigenvalues of nonlinear odd operators along with variational and topological methods. In addition, Vorovich investigated one of the variants of the analytic method of Liapunov and Schmidt [2], and, as a concrete application, examined the problem of postbuckling behavior of a plate in the form of a circular annulus under small transverse loads and for various boundary conditions [7].

The method of [8] will be applied here to examine the problems described at the beginning of this paper. This differs somewhat from the method of Vorovich, although it is in essence very similar to it. Later it will be shown that if the values of p_0, q_0, $\varepsilon = 0$ are critical, then the vicinity of the point $p = p_0$, $q = q^0$, $\varepsilon = 0$ is divided in two parts by some surface (the difurcation surface), such that there exists a single small solution in one of these parts and three small solutions in the other. It is assumed here that zero is a simple eigenvalue of the corresponding linearized boundary value problem.

Approximate expressions are given for all the small solutions.

1. Formulation of the problem and reduction to a single operator equation. Let the plate occupy the bounded region Ω with sufficiently smooth boundary Γ. On the plate there act small normal loads and external boundary tractions having normal and shear components of the form

$$\sigma_n|_\Gamma = p\cos^2\theta + q\sin^2\theta,\ \sigma_\tau|_\Gamma = \frac{1}{2}(p-q)\sin 2\theta$$

where p and q are the components of the external tractions in the x and y directions, and θ is the angle which the normal n makes with the x axis at the corresponding point of Γ. The nonlinear system of von Kármán equations can then be written in the form

$$\Delta^2 F + \frac{1}{2}[w,\ w] = 0,\ \varkappa^2\Delta^2 w - [w,\ F] - \varepsilon Q = 0 \tag{1.1}$$

$$[w,\ F] = w_{xx}F_{yy} + w_{yy}F_{xx} - 2w_{xy}F_{xy} \tag{1.2}$$

with the boundary conditions

$$F|_\Gamma = 0, \qquad \frac{\partial F}{\partial n}\Big|_\Gamma = 0 \tag{1.3}$$

In addition, it is assumed that the plate is either rigidly clamped at its edge

$$w|_\Gamma = 0, \qquad \frac{\partial w}{\partial n}\Big|_\Gamma = 0 \tag{1.4}$$

or else is simply supported these, i. e.

$$w|_\Gamma = 0, \qquad \left[\frac{\partial^2 w}{\partial n^2} + \frac{\sigma}{\rho}\frac{\partial w}{\partial n}\right]_\Gamma = 0 \tag{1.5}$$

Equations (1.1) – (1.5) are written in dimensionless form, with

$$w = \frac{W}{d},\quad F = \frac{\Phi}{Ed^2},\quad x = \frac{x_1}{d},\quad y = \frac{y_1}{d},\quad \varkappa^2 = \frac{h^2}{12(1-\sigma^2)d^2}$$

$$\varepsilon Q = \frac{q_1 d}{Eh},\quad 0 < \sigma < 0.5$$

where W is the deflection of the plate, Φ is the stress function, d is a characteristic diameter of the region Ω, x_1 and y_1 are rectangular coordinates, q_1 is the intensity of the normal loading, h is the thickness of the plate, E is Young's modulus, σ is Poisson's ratio, and ρ is the radius of curvature of the boundary at a point of Γ.

The aim is to investigate the problem (1.1) – (1.5) in the neighborhood of the values of the parameters p_0 and q_0. To do this, we set

$$p = p_0 + \lambda,\ q = q_0 + \mu \tag{1.6}$$

We now introduce the Banach spaces E_1 and E_2 of two-dimensional columns of functions

$$u(x,\ y) = \left\|\begin{matrix} w(x,\ y) \\ F(x,\ y)\end{matrix}\right\|$$

the components of which, w and F, belong to the space of S. L. Sobolev $W_\alpha^4(\Omega)$ $(\alpha > 1)$ and satisfy the boundary conditions (1.4) or (1.5) and (1.3), respectively.

Further, let E_2 be the space of the columns

$$h(x,\ y) = \left\|\begin{matrix} v(x,\ y) \\ \Phi(x,\ y)\end{matrix}\right\|$$

the components of which belong to $L_\alpha(\Omega)$ $(\alpha > 1)$. In these spaces the problem can be written in the form of a single functional equation if Eq. (1.6) is kept in mind

$$Bu = \varepsilon\varphi + \lambda Au + \mu Cu + Du \tag{1.7}$$

where B, A and C are linear operators from E_1 into E_2

$$B=\left\|\begin{matrix}\varkappa^2\Delta^2-p_0\partial^2(\cdots)/\partial y^2-q_0\partial^2(\cdots)/\partial x^2 & 0\\ 0 & \Delta^2\end{matrix}\right\|,\qquad A=\left\|\begin{matrix}\partial^2(\cdots)/\partial y^2 & 0\\ 0 & 0\end{matrix}\right\|$$

$$C=\left\|\begin{matrix}\partial^2(\cdots)/\partial x^2 & 0\\ 0 & 0\end{matrix}\right\|,\qquad \varphi=\left\|\begin{matrix}Q\\ 0\end{matrix}\right\| \tag{1.8}$$

Here φ is an element of E_2, and $D(u)$ is a quadratic operator from E_1 into E_2 generated by the symmetric bilinear expression

$$2D(u_1,\ u_2)=\left\|\begin{matrix}[w_1,\ F_2]+[w_2,\ F_1]\\ -[w_1,\ w_2]\end{matrix}\right\|,\qquad D(u)=D(u,\ u)=\left\|\begin{matrix}[w,\ F]\\ -[w,\ w]\end{matrix}\right\| \tag{1.9}$$

The possibility of expressing the problem in the form of Eq. (1.7) in the above spaces follows from the a priori estimates for the equations of the theories of plates and shells established in papers by Vorovich [9] and Morozov [10].

2. Continuation of the solutions. We shall first examine the simplest case in which the operator B has a bounded inverse B^{-1}. We write Eq.

$$\varkappa^2\Delta^2 w-p_0 w_{yy}-q_0 w_{xx}=0,\qquad \Delta^2 F=0 \tag{2.1}$$

with the boundary conditions (1.3) and (1.4)(or (1.5)). It follows from (2.1) that $F\equiv 0$. Thus the entire investigation reduces to a study of the first equation of (2.1) with the boundary conditions (1.4) (or (1.5)).

Let p_0 and q_0 be such that this problem does not lie on the spectrum. This will be the case, in particular, if $p_0>0$ and $q_0>0$. Physically, this means that the plate is being stretched by the external tractions applied to its edge. Under these conditions the operator B^{-1} exists and is bounded. It follows from the theorem on implicit operators [11] that there exists a unique small solution $u=u(\varepsilon,\lambda,\mu)$ satisfying the condition $u(0,0,0)=0$ this solution can be found [12] in the form of a series in integral powers of the parameters ε, λ and μ.

Now let p_0 and q_0 be such that the problem (2.1) lies on the spectrum. Then it has a finite number of linearly independent solutions $f_1, f_2, \ldots, f_n$. From the fact that the formally adjoint problem coincides with (2.1) it follows that the condition

$$\iint_\Omega v(x,\ y)\,f_i(x,\ y)\,dx\,dy=0\qquad (i=1,\ 2,\ \ldots n) \tag{2.2}$$

is necessary and sufficient for the solvability of the inhomogeneous equation $Bu=v$

In order to investigate Eq. (1.7) we may now form the Liapunov-Schmidt bifurcation equation [12]. In this case it has the form of a system of n equations with the n unknowns $\xi_1, \xi_2, \ldots, \xi_n$ and the parameters λ, μ and ε

$$L^{(s)}(\xi_1,\xi_2,\ldots,\xi_n\,\varepsilon,\ \lambda,\ \mu)=0,\quad (s=1,2,\ldots,n) \tag{2.3}$$

The study of this system in general form presents great difficulties. The presence of more than one parameter means that instead of a point of bifurcation as in the case of a single parameter, surfaces (or curves) of difurcation appear. These split the neighborhood of the point $\varepsilon=\mu=\lambda=0$ into parts; the crossing of these surfaces correspond to the creation (or disappearance) of new pairs of solutions.

3. Investigation of the buckling phenomenon for $n=1$. We shall now concentrate on the case $n=1$. Here the equation of bifurcation (2.3) has the form

$$L(\varepsilon, \lambda, \mu, \xi) \equiv \sum_{i+j+k\geqslant 0} L_{ijl0}\varepsilon^i\lambda^j\mu^k + \sum_{l=1}^{\infty}\xi^l \sum_{i+j+k\geqslant 0} L_{ijkl}\varepsilon^i\lambda^j\mu^k = 0 \tag{3.1}$$

(the scripts indicate what powers of ε, λ, μ and ξ the coefficient multiplies).

Let us calculate the leading coefficients. Arguing in the same way as in Section 4.3 of the paper [12], and using (2.2) for $n = 1$, we obtain

$$a_0 \equiv L_{1000} = \iint_\Omega Qf\,dx\,dy, \qquad \iint_\Omega f^2 dx\,dy = 1 \tag{3.2}$$

where f is the nontrivial solution of the problem (2.1) satisfying the second condition in (3.2). The coefficients $L_{0100} = L_{0010} = 0$, since the equation has no constant terms containing λ and μ

$$a_1 \equiv L_{0101} = \iint_\Omega \frac{\partial^2 f}{\partial y^2} f\,dx\,dy = -\iint_\Omega \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy < 0 \tag{3.3}$$

In exactly the same way

$$a_2 \equiv L_{0011} = -\iint_\Omega \left(\frac{\partial f}{\partial x}\right)^2 dx\,dy < 0 \tag{3.4}$$

To find the first nonzero coefficient L_{000l} $(l \geqslant 2)$ we use the method proposed in [12]. We introduce the operator B_1 defined for $u \in E_1$ by Formula

$$B_1 u = \left\| \begin{matrix} \Delta^2 w - p_0 \dfrac{\partial^2 w}{\partial y^2} - q_0 \dfrac{\partial^2 w}{\partial x^2} + \iint_\Omega wf\,ds\,dt\,f(x, y) \\ \Delta^2 F \end{matrix} \right\| \tag{3.5}$$

According to the generalized Schmidt lemma, the operator B_1 has a bounded inverse operator. We examine Eq.

$$B_1 u = D(u) + \xi\varphi_1, \quad \varphi_1 = \left\| \begin{matrix} f \\ 0 \end{matrix} \right\| \tag{3.6}$$

where ξ is an arbitrary small parameter. The solution of Eq. (3.6) can be found in the form of an infinite series which converges with respect to the metric of $W_\alpha^{(4)}(\Omega)$

$$u(\xi) = u_1\xi + u_2\xi^2 + u_3\xi^3 + \cdots \tag{3.7}$$

The coefficients L_{000l} $(l \geqslant 2)$ can now be calculated from the relations

$$\sum_{l=2}^{\infty} L_{000l}\xi^l = \iint_\Omega D_1(u(\xi))\, f\,dx\,dy \tag{3.8}$$

where D_1 denotes the first component of D. Substituting (3.7) into (3.8) we obtain a system of recurrence relations which determine u_i $(i = 1, 2, \ldots)$

$$B_1 u_1 = \varphi_1, \quad B_1 u_2 = D(u_1), \quad B_1 u_3 = 2D\,(u_1, u_2), \ldots \tag{3.9}$$

By virtue of (3.5) and (3.2), we find from Eq. (3.9) that $u_1 = \varphi_1$. To find u_2 we first compute $Du_1 = D\varphi_1$. Using (1.9) we obtain

$$D(u_1) = \left\| \begin{matrix} 0 \\ -\tfrac{1}{2}[f, f] \end{matrix} \right\|$$

It follows from this by virtue of (3.8) that $L_{0002} = 0$. We shall now show that

$$a_3 \equiv L_{0003} = -\int_\Omega (\Delta F_2)^2\,dx\,dy < 0 \tag{3.10}$$

where F_2 is a certain function which will be defined below. We then find that

$$u_2 = \left\| \begin{matrix} 0 \\ F_2 \end{matrix} \right\|$$

where F_2 is the solution of the boundary value problem

$$\Delta^2 F_2 = -\tfrac{1}{2}[f,\, f], \qquad F_2|_\Gamma = \partial F_2/\partial n|_\Gamma = 0 \tag{3.11}$$

Using (1.9), we conclude that
$$2D\,(u_1 u_2) = \left\| \begin{matrix} [f,\, F_2] \\ 0 \end{matrix} \right\|$$

Now by applying (3.8) and (3.11) and integrating by parts, we obtain

$$a_3 = 2\iint_\Omega D_1(u_1, u_2)\, f dx\, dy = \iint_\Omega [f,\, F_2]\, f\, dx\, dy = \iint_\Omega F_2\, [f,\, f]\, dx\, dy = -\,2\iint_\Omega (\Delta F_2)^2\, dx\, dy$$

It follows from (3.10) that for all sufficiently small ε, λ and μ, the problem has precisely three complex solutions. We construct approximately the surface of bifurcation which, as will be clear later, splits the vicinity of the point $\varepsilon = \lambda = \mu = 0$ into two parts, in one of which there exists one solution, and in the other three. Two of the latter merge on the bifurcation surface.

The approximate equation of bifurcation has the form

$$a_0\varepsilon + a_1\lambda\xi + a_2\mu\xi + a_3\xi^3 + \ldots = 0 \tag{3.12}$$

4. The asymptotic behavior of small solutions. We first examine the simplest case in which $Q \equiv 0$. Then $L_{i000} = 0\ (i = 1,2,\ldots)$, and the bifurcation equation has ξ as a factor; $\xi = 0$ corresponds to the trivial solution of the problem. Cancelling ξ, we arrive at Eq.

$$a_1\lambda + a_2\mu + a_3\xi^2 + \ldots = 0 \tag{4.1}$$

It is clear from this that the neighborhood of the point $\lambda = \mu = 0$ is divided into two parts by the bifurcation curve (the approximate equation of which is $a_1\lambda + a_2\mu = 0$). Above the curve, Eq. (4.1) has no small solutions, but below it there are precisely two solutions. To the first approximation these solutions are given by Eqs.

$$\xi_{1,2} = \pm\,[\,-\,a_3^{-1}(a_1\lambda + a_2\mu)]^{1/2} \tag{4.2}$$

In this case, the problem (1.1) – (1.5) has, in addition to the trivial solution, two small nontrivial solutions which may be represented asymptotically in the form

$$w_{1,2} \approx \xi_{1,2}(\lambda,\, \mu)\, f(x,\, y), \quad F_{1,2} \approx 0 \tag{4.3}$$

We remark that the special case $p = q$ (i.e. $\lambda = \mu$) was considered in [4], where it was proved by a topological method that two new solutions arise as the critical value is crossed. We supplement these results by noting that in this case the solution can be found [12] in the form of a convergent series in powers of $(-\,\lambda)^{1/2}$, and that the first approximation is obtained from Eq. (4.2) with $\mu = \lambda$. Exactly the same conclusions can be drawn for the case in which p and q are connected by some relationship.

Let us now consider the case when $Q \not\equiv 0$. We shall limit ourselves here to values of Q such that $a_0 \neq 0$. We can conclude from Eq. (3.12) that the neighborhood of $\varepsilon = \lambda = \mu = 0$ is divided into two parts by the surface of bifurcation. In one of these parts there is a single solution; in the other there are three. In order to construct this surface approximately we form the eliminant $R(\varepsilon,\, \lambda,\, \mu)$ of the polynomials

$$P(\varepsilon,\ \lambda,\ \mu,\ \xi) = a_0\varepsilon + a_1\lambda\xi + a_2\mu\xi + a_3\xi^3, \qquad \partial P/\partial\xi = a_1\lambda + a_2\mu + 3a_3\xi^2$$

The equation of the discriminant surface, which coincides with the bifurcation surface $R(\varepsilon,\, \lambda,\, \mu) = 0$ has the form
$$4(a_1\lambda + a_1\mu)^3 + 27a_0^2 a_3 \varepsilon^2 = 0 \tag{4.4}$$

and is a cylindrical surface with a semicubic parabola as directrix.

We remark that in this case all the solutions of the problem are approximately representable by Expressions

$$w \sim \xi(\varepsilon, \mu, \lambda)\, f(x, y), \; F \sim 0$$

where ξ is a solution of the cubic equation

$$P(\varepsilon, \lambda, \mu, \xi) = 0$$

Let us examine the special case $p = kq$ (i.e. $\lambda = k\mu$). in greater detail. Then

$$P(\varepsilon, \lambda, \mu, \xi) \equiv a_0\varepsilon + b\mu\xi + a_3\xi^3 = 0, \qquad b = a_1k + a_2 \tag{4.5}$$

(We assume that if $k = \infty$, $b = a_1$ and in the succeeding calculations μ must be replaced by λ.) To investigate the solution for ξ asymptotically it is convenient to transform from the two small parameters ε and μ in Eq. (4.5) to a single parameter which takes on arbitrary real values. This can be done by introducing the new variables θ and η according to Eqs.

$$\xi = \left(\frac{a_0}{a_3}\varepsilon\right)^{1/3}\eta, \qquad \theta = b\mu\,(a_0\varepsilon)^{-2/3}a_3^{-1/3} \tag{4.6}$$

We then have from Eq. (4.5)

$$1 + \theta\eta + \eta^3 = 0 \tag{4.7}$$

The point A $(-{}^3/_2(2)^{1/3},\ {}^1/_2\,(4)^{1/3})$ of the curve representing the function $\eta = \eta(\theta)$ is a branch point. There exist three single-valued branches: $\eta = \eta_0(\theta)$, $\eta = \eta_+(\theta)$ and $\eta = \eta_-(\theta)$. The last two are defined to the left of the point A and merge at that point.

By defining $\eta_0(\theta)$, $\eta_-(\theta)$ and $\eta_+(\theta)$ with the requisite degree ot accuracy from (4.7), it is then possible with the aid of Eqs. (4.6) and (4.3) to find the small solutions of the problem (1.1) – (1.5).

We shall give without derivation, for example, the asymptotic behavior of η_0, η_- and η_+ for $\theta \to \pm\infty$ and for $\theta \to A - 0$. The first case corresponds the asymptotic behavior of the associated values of ξ away from a small strip bordering the curve of bifurcation, and the second corresponds to the asymptotic behavior near the bifurcation curve. From (4.6) and (4.7) we have

$$\eta_0(\theta) \sim -(-\theta)^{1/2}, \qquad \eta_+(\theta) \sim (-\theta)^{1/2}, \qquad \eta_-(\theta) \sim -1/2\theta$$

In exactly the same way for $\theta \to -\infty$ we find from (4.7)

$$\eta_0(\theta) \sim -(-\theta)^{1/2}, \qquad \eta_+(\theta) \sim (-\theta)^{1/2}, \qquad \eta_-(\theta) \sim -1/2\theta$$

Applying (4.6), we obtain for $\mu\varepsilon^{-2/3} \to -\infty$

$$\xi_{0,+}(\varepsilon, \mu) \sim \mp\,\mathrm{sign}\,(a_0\varepsilon)\,\frac{(-b\mu)^{1/2}}{a_3}, \qquad \zeta_-(\varepsilon, \mu) \sim \frac{a_0\varepsilon}{\mu}$$

We now construct the asymptotic solution near the curve of bifurcation. We carry out the transformation of variables

$$\theta = -3\cdot 2^{-2/3} - \tau, \quad \eta = 2^{-1/3} + \sigma$$

It is not difficult to obtain from (4.7) that

$$\sigma \sim \pm\,({}^1/_3\,\tau)^{1/2} \quad \text{for } \tau \to 0,\ \sigma \to 0$$

It follows from this that

$$\eta_\pm(\theta) \sim 2^{-1/3} \pm [-3\cdot 2^{-2/3} - \theta]^{1/2} \quad \text{for } \theta \to -3\cdot 2^{-2/3}$$

Finally, by virtue of (4.6) we obtain

$$\xi_\pm(\varepsilon, \mu) \sim (a_0\varepsilon / a_3)^{1/3}\{2^{-1/3} \pm [-3\cdot 2^{-2/3} - b\mu\,(a_0\varepsilon)^{-2/3}a_3^{-1/3}]^{1/2}\}$$

$$\varepsilon_0(\varepsilon, \mu) \sim (4a_0\varepsilon / a_3)^{1/3} \quad \text{and} \quad \mu\varepsilon^{-2/3} \to -3\cdot 2^{-2/3}\, a^{1/3}a^{2/3}h^{-1}$$

5. Example. A circular, symmetrically loaded plate. Transforming to plane polar coordinates (r, θ) and carrying out the following transformation of variables in (1.1) – (1.4):

$$\frac{dw}{dr} = \vartheta, \qquad \frac{1}{r}\frac{dF}{dr} = p + \psi, \qquad \int_0^r Q(s)\, s ds = Q_1(r)$$

we obtain the equations of a circular plate in the case of radial symmetry

$$A\psi - {}^1/_2\, \vartheta^2 = 0, \qquad \varkappa^2 A\vartheta + p_0\vartheta = -\varepsilon Q_1(r) - \lambda\vartheta - \vartheta\psi$$

$$A(\cdots) \equiv -r\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r(\cdots), \qquad 0 \leqslant r \leqslant 1 \tag{5.1}$$

with the boundary conditions

$$[\vartheta / r]_{r=0} < \infty, \ [\psi / r]_{r=0}, < \infty \ \psi(1) = 0, \quad \vartheta(1) = 0 \tag{5.2}$$

Here we take the space of two-dimensional columns of functions having elements in the space $C_2(0,1)$ which satisfy the boundary conditions (5.2) as our space E_1. For E_2 we take the space $C_0\,(0,1)$.

With the aid of matrix notations analogous to those given in Section 1, the problem is rewritten in the form of Eq. (1.7). The critical values of the parameter p_0 and the corresponding eigenfunctions are determined from the boundary value problem

$$\frac{d^2\vartheta}{dr^2} + \frac{1}{r}\frac{d\vartheta}{dr} - \left(p_0 + \frac{1}{r^2}\right)\vartheta = 0, \qquad \vartheta(1) = \vartheta(0) = 0 \tag{5.3}$$

The problem (5.3) has a countable set of negative eigenvalues $p_{0k} = -\alpha_k^2$, where the α_k's are the positive roots of the Bessel function $J_1(r)$. All the p_{0k} are simple eigenvalues and correspond to the eigenfunctions $J_1(\alpha_k r)$. The coefficients of the equation of bifurcation (4.1) may be computed as follows $(\lambda = \mu)$

$$a_0 = \int_0^1 Q_1 J_1(\alpha_k r)\, dr, \qquad a_1 = a_2 = -\frac{1}{2}\int_0^1 J_1^2(\alpha_k r)\, dr < 0$$

$$a_3 = -\int_0^1 \left[\left(\frac{d\vartheta}{d\rho}\right)^2 + \frac{\vartheta^2}{\rho^2}\right] d\rho < 0 \qquad \left(\vartheta = r\int_r^1 \frac{dt}{t^3}\int_0^t J_1^2(s)\, s\, ds\right)$$

Exactly the same conclusions are obtained as in Section 4.

BIBLIOGRAPHY

1. Polubarinova-Kochina, P. Ia., On the problem of stability of plates. PMM Vol. 3, №1, 1936.
2. Vorovich, I. I., On some direct methods in the nonlinear theory of shallow shells. PMM Vol. 20, №4, 1956.
3. Friedrichs, K. and Stoker, J., The Non-linear Boundary Value Problem of the Buckled Plate. Amer. J. Math., Vol. 63, №4, 1941.
4. Berger, M. and Fife, P., On von Kármán's equations and the buckling of a thin elastic plate. Bull. Amer. Math. Soc., Vol. 72, №6, 1966.
5. Krasnosel'skii, M. A., Topological Methods in the Theory of Nonlinear Integral Equations. Moscow, Gostekhizdat, 1956 (English translation published by Pergamon Press).
6. Vorovich, I. I. Some problems of stability of shells in the large. Dokl. Akad. Nauk SSSR, Vol. 122, №1, 1958.
7. Vorovich, I. I., Postbuckling behavior of a circular plate. Uch. zap. Rostov. Univ. (Scientific Memoirs of Rostov Univ.), Vol. 32, №4, 1955.

8. Trenogin, V. A., The equation of bifurcation and Newton's polygon. Dokl. Akad. Nauk SSSR, Vol. 131, №5, 1960.
9. Vorovich, I. I., On the existence of solutions in the nonlinear theory of shells. Dokl. Akad. Nauk SSSR, Vol. 117, №2, 1957.
10. Morozov, N. F., On the nonlinear theory of thin plates. Dokl. Akad. Nauk SSSR, Vol. 114, №5, 1957.
11. Liusternik, L. A. and Sobolev, V. I., Elements of Functional Analysis. Moscow, Fizmatgiz, 1966.
12. Vainberg, M. M. and Trenogin, V. A., The methods of Liapunov and Schmidt in the theory of nonlinear equations and their subsequent development. Usp. matem. nauk, Vol. 17, №2, 1962.

Translated by A. R. R.

A DYNAMIC SYSTEM WITH A DISCONTINUOUS CHARACTERISTIC

PMM Vol. 32, №4, 1968, pp. 728-734

N. N. SKRIABIN
(Gor'kii)

(Received April 8, 1968)

We consider a differential equation (which finds practical applications), containing a function with finite discontinuities, and its derivative. Since the equation is not defined on the lines of discontinuity, additional definitions are constructed for various cases.

The supplementary definition schemes supply the information needed for a complete qualitative analysis of the system. Examples of such analysis are given for two different characteristics corresponding to two differing cases of supplementary definition.

Equation
$$\varphi^{\cdot\cdot} + 2h[1 - bF'(\varphi)]\ \varphi^{\cdot} + F(\varphi) = \Omega, \quad F(\varphi + 2\pi) = F(\varphi) \tag{1}$$

encountered in practice (*) was studied often (see e.g. [1] and [2]) for the case when $F(\varphi)$ has continuous characteristics.

Here we propose a method of investigation of (1), when the characteristic exhibits finite discontinuities.

Let $\varphi = \varphi_n$ be one of the points of discontinuity. The system
$$\varphi^{\cdot} = y, \quad y^{\cdot} = \Omega - F(\varphi) - 2h[1 - bF'(\varphi)]y \tag{2}$$
equivalent to (1) is not defined on the line $\varphi = \varphi_n$. Therefore, when the representative

---

*) When $b > 0$, Eq. (1) represents the equation of the phase automatic frequency control (afc) with an integrating filter with delay; when $b < 0$, Eq. (1) is the equation of automatic control with a proportional integrating filter without delay [1].